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OSCULATION OF TWO NONLINEARLY ELASTIC BODIES

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The problem of osculation of two solids S_1 and S_2 similar in shape to a half-plane and made of a nonlinearly elastic harmonic-type material is investigated [1]. The contact area is assumed to be free of friction. An exact solution is obtained.

1. We consider physical regions S_1 and S_2 with boundaries of close-to-linear shape. After deformation, they come into contact along the common portion L of their respective boundaries L_1 and L_2. The contact of the bodies is accomplished by external forces, whose principal vector is $P_0 = X + iY$ (P_0 is a known constant). The contact region L is assumed to consist of a finite number of segments of the real axis ox: $L = [a_1b_1] + \ldots + [a_nb_n]$. Suppose that S_1 and S_2 occupy the lower and upper half-planes of the plane of the variable $z = x + iy$ [2]. Quantities referring to S_1 and S_2 will be identified by subscripts 1 and 2, respectively. Stresses and rotations are absent for these bodies at infinity.

The boundary conditions of the problem are [3]

$$v_1^- - v_2^+ = f(\overset{*}{x}),\ T_1(x) = T_2(x) = 0,\ N_1(x) = N_2(x) = N(x) \text{ on } L, \tag{1.1}$$

On the remaining portions of the boundaries that are free of external actions

$$N = 0,\ T = 0. \tag{1.2}$$

Here N and T are the normal and the tangential stresses; v is the normal elastic displacement; $f(\overset{*}{x}) = f_2(\overset{*}{x}) - f_1(\overset{*}{x})$ is a function specified on the deformed contact line; f_1 and f_2 characterize the configuration of the compressed bodies after deformation. It will be recalled that $\overset{*}{x} = x + u$, $u = u(x)$ is the horizontal elastic displacement of the points of the line L. We will assume that $f'(\overset{*}{x}) \in H(L)$.

The solution makes use of a complex representation of the fields of elastic elements for a half-plane in terms of two functions $\varphi(z)$ and $\psi(z)$ of the complex argument $z = x + \iota y$ which are analytic in that half-plane [4]:

$$X_x + Y_y + 4\mu = \frac{(\lambda+2\mu)\, q\Omega(q)}{\sqrt{J}},\quad Y_y - X_x - 2\iota X_y = -\frac{4(\lambda+2\mu)}{\sqrt{J}}\,\frac{\Omega(q)}{q}\,\frac{\partial z^*}{\partial z}\,\frac{\partial z^*}{\partial \bar{z}}; \tag{1.3}$$

$$u + \iota v = \frac{\mu}{\lambda+2\mu}\int \varphi'^2(z)\,dz + \frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{\varphi(z)}{\overline{\varphi'(z)}} + \overline{\psi(z)}\right] - z; \tag{1.4}$$

$$\frac{\partial z^*}{\partial z} = \frac{\mu}{\lambda+2\mu}\varphi'^2(z) + \frac{\lambda+\mu}{\lambda+2\mu}\,\frac{\varphi'(z)}{\overline{\varphi'(z)}},\quad \frac{\partial z^*}{\partial \bar{z}} = -\frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{\varphi(z)\,\overline{\varphi''(z)}}{\overline{\varphi'^2(z)}} - \overline{\psi'(z)}\right], \tag{1.5}$$

where

$$z^* = z + u + \iota v;\quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right);\quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right);$$

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$$\sqrt{J}=\frac{\partial z^*}{\partial z}\frac{\overline{\partial z^*}}{\partial \bar{z}}-\frac{\partial z^*}{\partial \bar{z}}\frac{\overline{\partial z^*}}{\partial z},\quad q=2\left|\frac{\partial z^*}{\partial z}\right|,\quad \Omega(q)=q-\frac{2(\lambda+\mu)}{\lambda+2\mu};\tag{1.6}$$

λ and μ are elastic Lamé constants. At sufficiently large $|z|$, we have the representations [4]

$$\varphi(z)=-\frac{(\lambda+2\mu)(X+\iota Y)}{4\pi\mu(\lambda+\mu)}\ln z+z+o(1)+\text{const};\tag{1.7}$$

$$\psi(z)=\frac{(\lambda+2\mu)(X-\iota Y)}{2\pi\mu(\lambda+\mu)}\left[\frac{1}{2\varphi'(z)}-1\right]\ln z+o(1)+\text{const}.\tag{1.8}$$

Besides, $\varphi'(z)\neq 0$ everywhere in the closed region considered here.

By virtue of the second relation of (1.1), which expresses the absence of tangential stresses on L, and from (1.2), we write on the basis of (1.3) and (1.5)

$$\overline{\varphi_\iota(x)}\,\varphi_\iota''(x)-\varphi_\iota'^2(x)\,\psi_\iota(x)=0 \text{ on } \Gamma\quad (\iota=1,2)\tag{1.9}$$

($\Gamma=L_1\cup L_2\cup L$). By virtue of (1.9), from (1.3), (1.5), and (1.6) we write

$$N_\iota(x)=\frac{2\mu_\iota(\lambda_\iota+\mu_\iota)\left[\left|\varphi_\iota'^2(x)\right|-1\right]}{\lambda_\iota+\mu_\iota+\mu_\iota\left|\varphi_\iota'^2(x)\right|}\text{ on }\Gamma.\tag{1.10}$$

Proceeding from (1.1), (1.2), and (1.10), we formulate the conditions

$$\left|\varphi_\iota'(x)\right|=\left[\frac{\lambda_\iota+\mu_\iota}{\mu_\iota}\,\frac{2\mu_\iota+N(x)}{2(\lambda_\iota+\mu_\iota)-N(x)}\right]^{1/2}=f_\iota(x)\text{ on }L,\qquad \left|\varphi_\iota'(x)\right|=1\text{ on }\Gamma\diagdown L.\tag{1.11}$$

From (1.11), taking into account (1.7), we find the expressions

$$\varphi_1(z)=\exp\left[-\frac{1}{\pi\iota}\int_L\frac{F_1(x)\,dx}{x-z}\right],\quad z\in S_1;\tag{1.12}$$

$$\varphi_2'(z)=\exp\left[\frac{1}{\pi\iota}\int_L\frac{F_2(x)\,dx}{x-z}\right],\qquad z\in S_2,\tag{1.13}$$

where

$$F_\iota(x)=\frac{1}{2}\ln\left[\frac{\lambda_\iota+\mu_\iota}{\mu_\iota}\,\frac{2\mu_\iota+N(x)}{2(\lambda_\iota+\mu_\iota)-N(x)}\right]\quad(\iota=1,2)\tag{1.14}$$

are functions on L which are as yet unknown. We assume that $F_\iota(x)\in H$. We then differentiate (1.4) with respect to x. In the resulting relation we take into account (1.9):

$$1+u_j'+\iota v_j'=\varphi_j'^2(x)\left[\frac{\mu_j}{\lambda_j+2\mu_j}+\frac{\lambda_j+\mu_j}{\lambda_j+2\mu_j}\,\frac{1}{\left|\varphi_j'^2(x)\right|}\right]\text{ on }L\ (j=1,2).\tag{1.15}$$

We proceed in (1.15) to conjugate values, take the logarithms of the initial and resulting equations, and subtract one from the other:

$$\ln\left[\varphi_1'^{-2}(x)\,\varphi_2'^{-2}(x)\right]-\ln\left[\overline{\varphi_1'^{-2}(x)}\,\overline{\varphi_2'^{-2}(x)}\right]=2\iota\left[\operatorname{arctg}v_1'^{-}(\overset{*}{x})-\operatorname{arctg}v_2'^{-}(\overset{*}{x})\right]\text{ on }L\tag{1.16}$$

(differentiation is carried out with respect to the arguments in parentheses). The left-hand side of this equality, subject to the constraint

$$\ln\varphi_\iota'^{-}(\infty)=0\tag{1.17}$$

is a univalent function.

Now, by virtue of the familiar Sokhotskii–Plemelj relations we find the boundary values of the functions (1.2) and (1.3) on L and introduce the resulting expressions into the left-hand side of (1.16). Now, from the first condition of (1.1), after some manipulation, we write

$$\int_L \frac{F(x)\,dx}{x-x_0} = \frac{\pi}{2}\operatorname{arctg}\frac{f_1'(\overset{*}{x}) - f_2'(\overset{*}{x})}{1 + f_1'(\overset{*}{x}) f_2'(\overset{*}{x})} = g(\overset{*}{x}) = g(x + u(x)) = \delta(x), \tag{1.18}$$

where

$$F(x) = \frac{1}{2}\ln\left[\frac{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}{\mu_1\mu_2}\,\frac{(2\mu_1 + N(x))(2\mu_2 + N(x))}{(2(\lambda_1+\mu_1) - N(x))(2(\lambda_2+\mu_2) - N(x))}\right]. \tag{1.19}$$

The right-hand side of (1.18) is assumed to be a known function of the variable x. We denote by α the angle formed by a straight tangent drawn at the point ($\overset{*}{x}$, f($\overset{*}{x}$)) to the line L_1 and the positive direction of the axis o$\overset{*}{x}$; we denote by β the angle formed by a tangent to L_2 at the point ($\overset{*}{x}$, $f_2(\overset{*}{x})$) and the same straight line:

$$\operatorname{tg}\alpha(\overset{*}{x}) = f_1'(\overset{*}{x}), \quad \operatorname{tg}\beta(\overset{*}{x}) = f_2'(\overset{*}{x}). \tag{1.20}$$

Now

$$g(\overset{*}{x}) = (\pi/2)[\alpha(\overset{*}{x}) - \beta(\overset{*}{x})]. \tag{1.21}$$

Equality (1.18) with respect to the function F(x) on L is a characteristic singular integral equation of the first kind. The general solution of this equation of class h_0 (a solution not bounded at the endpoints of the contact line) appears as follows (this class corresponds to the index κ = n) [5]

$$F(x_0) = \Phi(x_0)/\sqrt{(x_0 - a_1)(x_0 - b_1)\ldots(x_0 - a_n)(x_0 - b_n)}, \tag{1.22}$$

where

$$\Phi(x_0) = -\frac{1}{2\pi}\int_L \frac{\sqrt{(x-a_1)(x-b_1)\ldots(x-a_n)(x-b_n)}\,\delta(x)\,dx}{x - x_0} + P(x_0), \tag{1.23}$$

$P(x_0) = C_1 x_0^{n-1} + C_2 x_0^{n-2} + \ldots + C_n$ is an arbitrary polynomial of degree not greater than $n - 1$. Note that in these relations the constants a_k and b_k ($k = 1, 2, \ldots, n$) are assumed to be known. The constants $C_1, C_2, \ldots, C_n$ are found from (1.7) and additional problem conditions.

For the solution of the class $h(a_1, b_1, \ldots, a_n, b_n)$ (the solution bounded in the neighborhood of the above-mentioned points), which corresponds to the index $\kappa = -n$ (the solution becomes zero at those points) the relations

$$\Phi(a_\iota) = 0,\ \Phi(b_\iota) = 0,\ \iota = 1, 2, \ldots, n \tag{1.24}$$

must take place. From these equalities the constants a_1 and b_1 can be calculated.

After defining from (1.18) F(x) on L, we obtain from (1.19) the desired function N(x) on the same region:

$$N(x) = \frac{-h + \sqrt{h^2 - 4\eta\gamma}}{\eta}, \tag{1.25}$$

where

$$h = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{\exp(2F)}{\lambda_1+\mu_1} + \frac{\exp(2F)}{\lambda_2+\mu_2}, \quad \eta = \frac{1}{\mu_1\mu_2} - \frac{\exp(2F)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}, \quad \gamma = 1 - \exp(2F). \tag{1.26}$$

The formula becomes particularly simple when both contacting bodies are made of the same material ($\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu$):

$$N(x) = \frac{2\mu\,[\exp(F(x)) - 1]}{1 + \frac{\mu}{\lambda+\mu}\exp(F(x))}. \tag{1.27}$$

After defining N(x) in this fashion and subtituting the resulting values into (1.14) and then substituting the results into (1.12) and (1.13), we can write the functions $\varphi_1(z)$ and

$\varphi_2(z)$. From expressions obtained from (1.8), (1.9), and (1.17) we derive by the ordinary procedure the potentials $\psi_1(z)$ and $\psi_2(z)$. Now we can calculate from (1.3)-(1.6) the stresses and displacements on any portion of the region investigated.

The principal problem of the theory of contact interactions is finding the dimensions of the contact area (if not known in advance) and the contact stresses on that area. The following two examples illustrate the realization of this method, which eliminates the flaws of the respective problems in the linear theory.

2. Example 1. We consider a contact area where the endpoints of the contact region are unknown and the contact area is a linear segment $[ab]$ parallel to the axis $o\overset{*}{x}$, where $n = 1$ and $\delta(x) = 0$.

The solution of (1.19) sought for, according to (1.20)-(1.23), appears as

$$F(x) = C/\sqrt{(x-a)(b-x)}, \tag{2.1}$$

where C is an arbitrary real constant. For determining this contact we introduce (2.1) into (1.12) and (1.13) and observe the asymptotic behavior of the resulting expressions at large $|z|$. Now, taking into account (1.7), we obtain

$$C = \frac{P_0}{4\pi}\left[\frac{\lambda_1 + 2\mu_1}{\mu_1(\lambda_1+\mu_1)} + \frac{\lambda_2 + 2\mu_2}{\mu_2(\lambda_2+\mu_2)}\right]. \tag{2.2}$$

Substituting (2.2) into (2.1), we obtain from (1.25) and (1.26) the value of N(x) on the contact area. For the same material at $a < x < b$ this expression is

$$N(x) = \frac{2\mu(\lambda+\mu)\left\{\exp\left[\frac{P_0(\lambda+2\mu)}{2\pi\mu(\lambda+\mu)}\frac{1}{\sqrt{(x-a)(b-x)}}\right]-1\right\}}{\lambda+\mu+\mu\exp\left[\frac{P_0(\lambda+2\mu)}{2\pi\mu(\lambda+\mu)}\frac{1}{\sqrt{(x-a)(b-x)}}\right]}. \tag{2.3}$$

A peculiarity of this formula (as opposed to the classical linear analog) is the fact that in the neighborhood of endpoints of the contact region it yields values of $2(\lambda + \mu)$, which are finite although fairly large [$\lim N(x) = 2(\lambda + \mu)$ as $x \to a^+$ or $x \to b^-$]. Besides, the distribution of contact stresses is substantially affected by the elastic properties of the contacting materials.

Table 1 gives the values of $N(x)/2\mu$ at different points of the contact area for the various ratios $P_0/2\mu$ (where μ is the shear modulus for a given material). The first values correspond to the linear theory; they are followed by the values for the nonlinear theory. (We assume that $a = -1$ and $b = 1$, and that a force P_0 is applied to the die at the symmetry center.)

As seen from Table 1, under the nonlinear model the contact normal stress approximately on the area $[-0.5; 0.5]$ under the die is smaller than in the classic linear case. By contrast, stress values outside this area tend to be larger. Remarkably, the difference between the linear and nonlinear results is not large. Even for $x = 0.99$ it does not exceed 5%.

In the general case (with different elastic materials) the situation is similar: The stress field on a closed contact area has no singularities. Therefore, we have constructed an exact solution for a problem in a closed region.

For determining the function $\varphi'(z)$ we take into account (2.1) and (2.2) in (1.12) and (1.13):

$$\varphi'(z) = \exp\left[(\lambda + 2\mu)P_0/(8\pi\mu(\lambda+\mu)\sqrt{(z-a)(b-z)})\right].$$

Example 2. We consider a single contact area where S_1 and S_2 are bounded by circles of radii R_1 and R_2, respectively; R_1 and R_2 are fairly large values (compared to the contact area). With an acceptable accuracy we can assume that

$$\delta(x) = \varepsilon x,\ \varepsilon = 1/R_1 + 1/R_2.$$

We should now find a solution of (1.18) bounded at the ends of the contact region subject to constraint (1.24). The calculation results at $\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu$, $R_1 = R_2 = R$; $a = -\ell$, $b = \ell$ are the following:

TABLE 1

P_0/μ	x						
	0	0,2	0,4	0,6	0,8	0,9	0,99
0,2	0,0318	0,0325	0,0347	0,0398	0,0531	0,0730	0,2257
	0,0304	0,0322	0,0346	0,0403	0,0542	0,0747	0,2373
0,4	0,0637	0,0650	0,0694	0,0796	0,1062	0,1460	0,4514
	0,0609	0,0648	0,0675	0,0812	0,1094	0,1517	0,4847
	0,0955	0,0974	0,1041	0,1194	0,1593	0,2190	0,6771
0,6	0,0905	0,0967	0,1040	0,1233	0,1656	0,2308	0,7182
	0,1272	0,1292	0,1388	0,1592	0,2124	0,2920	0,9028
0,8	0,1241	0,1261	0,1374	0,1656	0,2202	0,3108	0,9421

TABLE 2

λ/μ	x					
	0	0,2	0,4	0,6	0,8	1
1	0,1636	0,1628	0,1608	0,1498	0,1227	0
	0,1632	0,1612	0,1544	0,1424	0,1218	0
4	0,2087	0,2072	0,2008	0,1627	0,1553	0
	0,2040	0,2015	0,1931	0,1545	0,1541	0
	0,2335	0,2309	0,2247	0,2021	0,1723	0
16	0,2203	0,2176	0,2084	0,1932	0,1668	0

$$l = \sqrt{\frac{P_0 R}{\pi} \frac{\lambda + 2\mu}{\mu(\lambda + \mu)}}, \quad F(x) = \frac{\sqrt{l^2 - x^2}}{R},$$
$$N(x) = \frac{2\mu(\lambda + \mu)\left[\exp\left(\sqrt{l^2 - x^2}/R\right) - 1\right]}{\lambda + \mu + \mu \exp\left(\sqrt{l^2 - x^2}/R\right)}. \tag{2.4}$$

Taking into account (2.4) in (1.12) and (1.3), we obtain

$$\varphi'(z) = \exp\left(\left(\sqrt{l^2 - z^2} + \iota z\right)/R\right).$$

Hence, by virtue of (2.3), the contact stress goes to zero at the endpoints of the contact area.

Table 2 lists the values of $N(x)/\mu$ at the different points of the contact area for various λ/μ and at $P_0/\pi\mu = 1/10$. We see from (2.4) that the difference between the values $N(x)/\mu$ under nonlinear and linear theories is small. In particular, for these maximal stresses (at $x = 0$) it is at most 6%.

In a similar fashion we construct an effective solution for the case of two contact segments, both with known and unknown contact regions. Calculations indicate that in the former case contact stresses remain limited over the entire closed region, while in the latter case they become zero at the endpoints of the contact segments.

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